The lattice of varieties generated by residuated lattices of size up to 5

Peter Jipsen

Chapman University

Dedicated to Hiroakira Ono on the occasion of his 70th birthday

1 Introduction

Residuated lattices are the algebraic models of substructural logics, and have been studied since the late 1930's [9]. The varieties of residuated lattices and full Lambek algebras (FL-algebras) [7] include many well-known varieties of logic, such as the varieties of Boolean algebras, Gödel algebras, Heyting algebras, MV-algebras, Basic Logic algebras and involutive FL-algebras. These varieties contain many finite algebras and, apart from the variety of Boolean algebras, they all have infinitely many subvarieties. Recall that a *residuated lattice* is an algebra $(A, \wedge, \vee, \cdot, \mathbf{1}, \backslash, /)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, \mathbf{1})$ is a monoid, and for all $x, y, z \in A$ the equivalences $xy \le z \iff x \le z/y \iff y \le x\backslash z$ hold. An *FL-algebra* is a residuated lattice with an additional constant $\mathbf{0}$ that can denote any element of the algebra. For background about residuated lattices, FL-algebras and notation and terminology of universal algebra, we refer the reader to [3]. In particular, we consider residuated lattices as a subvariety of FL-algebras, defined by the identity $\mathbf{0} = \mathbf{1}$. Since all algebras considered here have lattice reducts, they have distributive congruence lattices. Moreover, the lattice $\Lambda(\mathsf{FL})$ of subvarieties of FL-algebras is a complete sublattice of the congruence lattice of the countably generated free FL-algebra, hence $\Lambda(\mathsf{FL})$ is a distributive lattice.

A variety $\mathcal{V}$ is called *finitely generated* if there is a finite set $\{A_1, A_2, \dots, A_n\}$ of finite algebras such that $\mathcal{V} = \mathsf{HSP}\{A_1, A_2, \dots, A_n\}$. Since any finite algebra is a subdirect product of finitely many finite subdirectly irreducible algebras, and since varieties are determined by there subdirectly irreducible algebras, we can assume that the generating algebras $A_1, \dots, A_n$ are subdirectly irreducible. Furthermore $\mathcal{V} = \mathsf{HSP}\{A_1\} \vee \cdots \vee \mathsf{HSP}\{A_n\}$, so if $\mathcal{V}$ is join-irreducible, then it is generated by a single finite subdirectly irreducible algebra. For congruence distributive varieties it follows from Jónsson's Lemma [5] that the converse also holds: if a variety is generated by a single finite subdirectly irreducible algebra then it is completely join-irreducible in the lattice of subvarieties and, in addition, it has only finitely many subvarieties, each itself finitely generated. Hence one concludes that the finitely generated varieties form an ideal in the lattice $\Lambda(\mathsf{FL})$.

By distributivity of $\Lambda(\mathsf{FL})$, the structure of this ideal is determined by the poset of join-irreducible varieties in it, i.e., there is a one-one correspondence between finitely generated varieties of FL-algebras and finite downsets in this

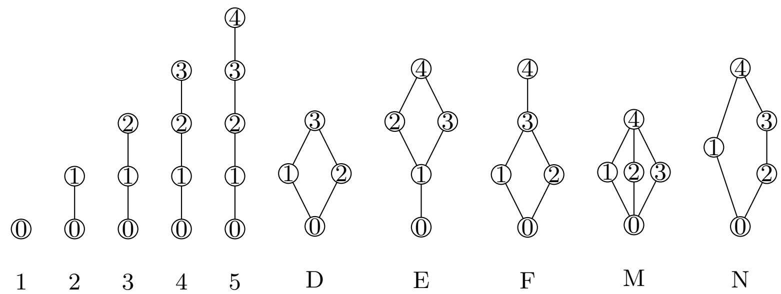

Fig. 1. Lattices of size up to 5

poset of join-irreducible varieties. By another application of Jónsson's Lemma, if A, B are finite subdirectly irreducible FL-algebras, then $\mathsf{HSP}\{A\} \subseteq \mathsf{HSP}\{B\}$ if and only if $A \in \mathsf{HS}\{B\}$. As a result, one obtains a description of the ideal of finitely generated varieties by computing the so-called HS-poset of finite subdirectly irreducible FL-algebras.

The aim of this note is to describe a small part of the bottom of this poset by enumerating the subdirectly irreducible residuated lattices of up to 5 elements and computing their subalgebras and homomorphic images (up to isomorphism). The results are summarised in several tables and diagrams below, together with a brief discussion of the familiar algebras within these tables and how they are organized. A longer list of finite residuated lattices is available at `www.chapman.edu/~jipsen/gap/rl.html`. An enumeration of commutative integral residuated lattices up to size 12 is at `vychodil.inf.upol.cz/order/`, [1].

2 Diagrams and tables

Rather than just providing lists of algebras, we would like to also give a view of the poset and arrange the algebras in a way that groups similar algebras together. We consider algebras to be similar if they satisfy the same identities that define specific well-known subvarieties of FL-algebras.

There are $1+1+3+20+149 = 174$ residuated lattices with 1,2,3,4,5 elements respectively. The lattice reducts of these algebras are listed in Figure 1. The n-element chain is simply denoted by n, and the remaining lattices are $D = 2 \times 2$, $E = 1 \oplus D$, $F = D \oplus 1$, M and N, where $\oplus$ denotes ordinal sum, and the last two lattices are the 5-element modular lattice and nonmodular lattice respectively, usually referred to as the diamond and the pentagon. For the 1,2,3,4,5-element chains there are exactly $1 + 1 + 3 + 15 + 84 = 104$ linearly ordered residuated lattices, and for the lattices D, E, F, M, N there are $5 + 20 + 11 + 8 + 26 = 70$ residuated lattices. Individual residuated lattices are denoted L_n, where $L \in \{1, 2, 3, 4, 5, D, E, F, M, N\}$ and n is an index that enumerates the algebras that have lattice L as reduct. So for example the three 3-element residuated lattices are $3_1, 3_2, 3_3$, and in the lists below they are the 3-element Wasjberg hoop (or MV-algebra if $\mathbf{0} = 0$), the 3-element Brouwerian algebra (or Gödel algebra if $\mathbf{0} = 0$) and the 3-element Sugihara algebra respectively.

RL var	FL var	Description
GBA	BA	(Generalized) Boolean algebras: $xx = x$ and WH, MV
WH	MV	Wajsberg hoops, MV-algebras: involutive BL-algebras
BH	BL	Basic hoops, Basic logic algebras: $x \wedge y = x(x\backslash y)$, CRRL, RFL_{ew}
RBr	GA	Representable Br-algs, Gödel algebras: s.i. are linear Br, HA
Br	HA	Browerian algebras, Heyting algebras: $xy = x \wedge y$
CRL	FL_e	Commutative RL, FL with the exchange rule: $xy = yx$
DRL	DFL	Distributive RL, FL: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
	InFL	Involutive RL, FL: $\mathbf{0}/(x\backslash\mathbf{0}) = x = (\mathbf{0}/x)\backslash\mathbf{0}$
IRL	FL_w	Integral RL: $x \leq \mathbf{1}$, FL with weakening: $\mathbf{0} \leq x \leq \mathbf{1}$
RRL	RFL	Representable RL, FL: s.i. algebras are linear

Table 1. Names of subvarieties

To fully specify a finite residuated lattice, it suffices to give its lattice reduct and a join-preserving monoid operation, since the residuals $\backslash, /$ are uniquely determined by this information, e.g., $z/y = \bigvee\{x \mid xy \leq z\}$. In the tables below, the monoid is presented as a transformation monoid, hence a residuated lattice is given by $\langle L_n, i$, list of transformations$\rangle$. Here L_n is the lattice reduct with a unique index, i is the element denoted by the identity constant $\mathbf{1}$, and each tuple $t = d_1 d_2 \ldots d_m$ of digits is a transformation $t : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, m\}$ where $t(0) = 0$ and $t(k) = d_k$ for $k = 1, \ldots, m$. Although it would suffice to give only a generating set of transformations, the tables give a complete set, except that the identity $123 \ldots m$ is omitted. This makes it very easy to construct the operation table for the monoid, since one can simply stack the transformations on top of each other, insert the identity transformation at row i, and add a row and column of zeros. For example, the residuated lattice $\langle E_{20}, 2, 1133, 1333, 1434\rangle$ has a monoid operation given by

$$\begin{array}{c|ccccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 3 & 3 \\
2 & 0 & 1 & 2 & 3 & 4 \\
3 & 0 & 1 & 3 & 3 & 3 \\
4 & 0 & 1 & 4 & 3 & 4
\end{array}$$

An FL-algebra is a residuated lattice with one extra constant $\mathbf{0}$, hence there are

$$1 \cdot 1 + 2 \cdot 1 + 3 \cdot 3 + 4 \cdot 15 + 5 \cdot 84 + 4 \cdot 5 + 5 \cdot (20 + 11 + 8 + 26) - 9 = 828$$

FL-algebras with up to 5 elements. The "-9" term is an adjustment since there are 9 residuated lattices among the 174 that have a nontrivial automorphism

RL var	FL var	Name, id, transformations	Sub	Hom
Triv	Triv	$\langle 1_1, 0\rangle$		
GBA	BA	$\langle 2_1, 1\rangle$		
WH	MV	$\langle 3_1, 2, 01\rangle$	2_1	
RBr	GA	$\langle 3_2, 2, 11\rangle$	2_1	2_1
CRRL	RInFL_e	$\langle 3_3, 1, 22\rangle$		
WH	MV	$\langle 4_1, 3, 001, 012\rangle$	2_1	
BH	BL	$\langle 4_2, 3, 011, 122\rangle$	3_1 3_2	3_1
		$\langle 4_3, 3, 111, 112\rangle$	3_1 3_2	2_1
RBr	GA	$\langle 4_4, 3, 111, 122\rangle$	3_2	3_2
CIRRL	RInFL_{ew}	$\langle 4_5, 3, 001, 022\rangle$	2_1	2_1
CIRRL	RFL_{ew}	$\langle 4_6, 3, 001, 002\rangle$	3_1	
IRRL	RFL_w	$\langle 4_7, 3, 001, 122\rangle$	2_1	
		$\langle 4_8, 3, 011, 022\rangle$	2_1	
CRRL	RInFL_e	$\langle 4_9, 1, 233, 333\rangle$	3_3	
		$\langle 4_{10}, 2, 113, 333\rangle$	2_1 3_3	3_3
CRRL	RFL_e	$\langle 4_{11}, 1, 223, 333\rangle$	3_3	
		$\langle 4_{12}, 2, 011, 133\rangle$		
		$\langle 4_{13}, 2, 111, 133\rangle$	3_3	2_1
RRL	RFL	$\langle 4_{14}, 2, 111, 333\rangle$		
		$\langle 4_{15}, 2, 113, 133\rangle$		
GBA	BA	$\langle D_1, 3, 101, 022\rangle$	2_1	2_1
CDRL	DInFL_e	$\langle D_{2,1}, 1, 202, 323\rangle$	3_3	
		$\langle D_{3,1}, 1, 213, 333\rangle$	3_3	
		$\langle D_{4,2}, 1, 233, 333\rangle$	3_3	
CDRL	DFL_e	$\langle D_5, 1, 222, 323\rangle$	3_3	

Table 2. Residuated lattices of size ≤ 4

group of size 2 (they are $D_1, E_1, E_3, E_5, F_1, F_3, M_3, M_4, M_8$, and in each case the automorphism switches the two nonidentity elements of the same height in the lattice). All other algebras in the list have no proper automorphisms. Note that the constant $\mathbf{0}$ and the least element 0 of each lattice are in general distinct, though they do coincide for FL_o-algebras (defined as FL-algebras where $\mathbf{0}$ is the bottom element). We do not list all FL-algebras separately in this note, but when we need to refer to a specfic one, we use the notation $L_{n,k}$ where k indicates which element of the lattice is the constant $\mathbf{0}$.

Subvarieties of FL-algebras use the same names as in [3]. In Table 1 we recall the relevant ones briefly to make this note more self-contained. Mostly the residuated lattice varieties (1st column) are obtained from the corresponding FL varieties (2nd column) by adding the equation $\mathbf{0} = \mathbf{1}$, but defining identities that refer to $\mathbf{0}$ are only applied to the FL varieties.

Intersections of these varieties are denoted by listing prefixes and/or subscript in alphabetical order, e.g., CDIRL and DFL_{ew}. We use the subvarieties to organize the algebras in the tables.

RL var	FL var	Name, id, transformations	Sub	Hom
WH	MV	$\langle 5_1, 4, 0001, 0012, 0123\rangle$	3_1	
BH	BL	$\langle 5_2, 4, 0011, 0122, 1233\rangle$	3_2 4_1	4_1
		$\langle 5_3, 4, 0111, 1222, 1223\rangle$	4_2 4_3	3_1
		$\langle 5_4, 4, 0111, 1222, 1233\rangle$	4_2 4_4	4_2
		$\langle 5_5, 4, 1111, 1112, 1123\rangle$	3_2 4_1	2_1
		$\langle 5_6, 4, 1111, 1122, 1233\rangle$	4_2 4_3 4_4	4_3
		$\langle 5_7, 4, 1111, 1222, 1223\rangle$	4_3 4_4	3_2
RBr	GA	$\langle 5_8, 4, 1111, 1222, 1233\rangle$	4_4	4_4
		$\langle E_1, 4, 1111, 1212, 1133\rangle$	3_2 D_1	3_2
Br	HA	$\langle F_3, 4, 1011, 0222, 1233\rangle$	3_2	D_1
CIRRL	RInFL$_{ew}$	$\langle 5_9, 4, 0001, 0012, 0113\rangle$	3_1 4_1	
		$\langle 5_{10}, 4, 0001, 0022, 0233\rangle$	3_1 4_5	3_1
CIRRL	RFL$_{ew}$	$\langle 5_{11}, 4, 0001, 0002, 0003\rangle$	4_6	
		$\langle 5_{12}, 4, 0001, 0002, 0013\rangle$	2_1	
		$\langle 5_{13}, 4, 0001, 0002, 0023\rangle$	4_1	
		$\langle 5_{14}, 4, 0001, 0002, 0033\rangle$	4_5	2_1
		$\langle 5_{15}, 4, 0001, 0112, 0113\rangle$	4_1	
		$\langle 5_{16}, 4, 0001, 0222, 0223\rangle$	3_1	2_1
		$\langle 5_{17}, 4, 0001, 0222, 0233\rangle$	3_2 4_5	3_2
		$\langle 5_{18}, 4, 0011, 0012, 1133\rangle$	3_1 3_2	3_1
		$\langle 5_{19}, 4, 0011, 0022, 1233\rangle$	4_2 4_6	4_6
		$\langle 5_{20}, 4, 0011, 0222, 1233\rangle$	3_2 4_5	4_5
		$\langle 5_{21}, 4, 1111, 1112, 1113\rangle$	4_3 4_6	2_1
		$\langle 5_{22}, 4, 1111, 1112, 1133\rangle$	3_2 4_5	3_2

Table 3.

If a residuated lattice can also be an involutive FL-algebra (with **0** redefined), then it appears in a section of the table that is labeled by a variety of involutive FL-algebras. The linear lattices and the pentagon have a unique dual automorphism, so for involutive FL-algebras with these lattice reducts the value of **0** is the image of **1** under this dual automorphism. However for D and M there are several dual automorphisms, and in this case we use names $D_{n,k}$ and $M_{n,k}$, where k gives the value of **0**. There are 3 residuated lattices of size ≤ 5 that can be involutive FL-algebras in two nonisomorphic ways. The algebras are listed in the table as $D_{3,1}$, $M_{1,1}$ and $M_{4,1}$ (also referred to as simply D_3, M_1, M_4), and alternative values of **0** give nonisomorphic InFL-algebras $D_{3,2}$, $M_{1,3}$ and $M_{4,2}$ (not listed in the tables, but they appear in Figure 4).

The tables of algebras also contain information about the HS-poset of the subdirectly irreducible residuated lattices. In the last two columns they list for each algebra the proper maximal nontrivial subdirectly irreducible subalgebras and the proper maximal nontrivial subdirectly irreducible homomorphic images respectively. So if the last column is empty, then the algebra is simple, and if the next-to-last column is also empty then the algebra is strictly simple and hence generates a variety that is an atom in the lattices of subvarieties. The results

RL var	FL var	Name, id, transformations	Sub	Hom
IRRL	RFL_w	$\langle 5_{23}, 4, 0001, 0002, 0113\rangle$	2_1	
		$\langle 5_{24}, 4, 0001, 0002, 0233\rangle$	2_1	
		$\langle 5_{25}, 4, 0001, 0002, 1133\rangle$	2_1	
		$\langle 5_{26}, 4, 0001, 0002, 1233\rangle$	4_7	
		$\langle 5_{27}, 4, 0001, 0012, 0013\rangle$	2_1	
		$\langle 5_{28}, 4, 0001, 0022, 0033\rangle$	2_1	
		$\langle 5_{29}, 4, 0001, 0022, 1233\rangle$	3_1	
		$\langle 5_{30}, 4, 0001, 0222, 1233\rangle$	3_2	2_1
		$\langle 5_{31}, 4, 0001, 1222, 1223\rangle$	3_1	
		$\langle 5_{32}, 4, 0001, 1222, 1233\rangle$	3_2 4_7	
		$\langle 5_{33}, 4, 0011, 0012, 0033\rangle$	2_1	
		$\langle 5_{34}, 4, 0011, 0012, 1233\rangle$	3_1 3_2	
		$\langle 5_{35}, 4, 0011, 0022, 0033\rangle$	4_8	
		$\langle 5_{36}, 4, 0011, 0022, 0233\rangle$	3_1	
		$\langle 5_{37}, 4, 0011, 0022, 1133\rangle$	3_1 3_2	
		$\langle 5_{38}, 4, 0011, 0222, 0233\rangle$	3_2	2_1
		$\langle 5_{39}, 4, 0011, 1222, 1233\rangle$	3_2 4_7	4_7
		$\langle 5_{40}, 4, 0111, 0222, 0223\rangle$	3_1	
		$\langle 5_{41}, 4, 0111, 0222, 0233\rangle$	3_2 4_8	
		$\langle 5_{42}, 4, 0111, 0222, 1233\rangle$	3_2 4_8	4_8
		$\langle 5_{43}, 4, 1111, 1112, 1233\rangle$	3_2 4_7	2_1
		$\langle 5_{44}, 4, 1111, 1122, 1133\rangle$	3_2 4_8	2_1
CRRL	RInFL_e	$\langle 5_{45}, 1, 2244, 3444, 4444\rangle$	4_9 4_{11}	
		$\langle 5_{46}, 1, 2344, 3444, 4444\rangle$	4_9	
		$\langle 5_{47}, 2, 1114, 1334, 4444\rangle$	3_3	3_3
		$\langle 5_{48}, 3, 0012, 0022, 2244\rangle$	4_{12}	
		$\langle 5_{49}, 3, 1114, 1124, 4444\rangle$	3_1 4_{10}	3_3

Table 4.

about subalgebras change if one considers FL-algebras, since in that case the constant **0** must also be in the subaglebra. This information is not indicated in the tables, but in the diagrams of the HS-posets dashed lines are used for ordering relations that hold for residuated lattices but not for FL_o-algebras.

Almost all the residuated lattices of size ≤ 5 are subdirectly irreducible. The only exceptions are 1_1, D_1 and E_1. Since there are so many subdirectly irreducible algebras, it is not practical to give a diagram of the HS-poset of all of them. Figure 2 shows the HS-poset of all subdirectly irreducible residuated lattices with ≤ 4 elements, Figure 3 shows it for linearly ordered integral residuated lattices with ≤ 5 elements, and Figure 4 shows it for involutive FL-algebras with ≤ 5 elements. Strictly speaking the trivial algebra (1_1) and the varieties in the upper half of these diagrams are not part of the HS-poset, but they indicate what equational properties are satisfied by the algebras that are below them. This helps with identifying some of the algebras in these diagrams. E.g. in Figure 3 we see that the algebras $2_1, 3_1, 4_1, 5_1$ are subdirectly irreducible

RL var	FL var	Name, id, transformations	Sub	Hom
CRRL	RFL_e	$\langle 5_{50}, 1, 2234, 3334, 4444\rangle$	4_{11}	
		$\langle 5_{51}, 1, 2234, 3344, 4444\rangle$	4_{11}	
		$\langle 5_{52}, 1, 2334, 3334, 4444\rangle$	4_{11}	
		$\langle 5_{53}, 1, 2444, 3444, 4444\rangle$	4_{9}	
		$\langle 5_{54}, 2, 0111, 1334, 1444\rangle$	4_{12}	
		$\langle 5_{55}, 2, 0111, 1344, 1444\rangle$	4_{12}	
		$\langle 5_{56}, 2, 1111, 1334, 1444\rangle$	4_{11} 4_{13}	2_1
		$\langle 5_{57}, 2, 1111, 1344, 1444\rangle$	4_{9} 4_{13}	2_1
		$\langle 5_{58}, 2, 1134, 3334, 4444\rangle$	4_{10} 4_{11}	4_{11}
		$\langle 5_{59}, 2, 1134, 3344, 4444\rangle$	4_{9} 4_{10}	4_{9}
		$\langle 5_{60}, 3, 0011, 0022, 1244\rangle$	4_{12}	
		$\langle 5_{61}, 3, 0011, 0122, 1244\rangle$		
		$\langle 5_{62}, 3, 0011, 0222, 1244\rangle$	3_3	2_1
		$\langle 5_{63}, 3, 0111, 1222, 1244\rangle$	4_{13}	3_1
		$\langle 5_{64}, 3, 0111, 1224, 1444\rangle$	2_1 4_{12}	4_{12}
		$\langle 5_{65}, 3, 1111, 1122, 1244\rangle$	4_{12}	2_1
		$\langle 5_{66}, 3, 1111, 1222, 1244\rangle$	4_{13}	3_2
		$\langle 5_{67}, 3, 1111, 1224, 1444\rangle$	4_{10} 4_{13}	2_1 4_{13}
		$\langle 5_{68}, 3, 1114, 1224, 4444\rangle$	3_2 4_{10}	4_{10}
CDIRL	DFL_{ew}	$\langle F_1, 4, 0001, 0002, 0003\rangle$	4_6	
		$\langle F_2, 4, 0001, 0222, 0223\rangle$	3_1	2_1
DIRL	DFL_w	$\langle E_2, 4, 0011, 1212, 0033\rangle$	2_1	
RRL	RFL	$\langle 5_{69}, 1, 2234, 3444, 4444\rangle$	4_{11}	
		$\langle 5_{70}, 1, 2244, 3344, 4444\rangle$	4_{11}	
		$\langle 5_{71}, 2, 1111, 1334, 4444\rangle$		
		$\langle 5_{72}, 2, 1114, 1334, 1444\rangle$		
		$\langle 5_{73}, 2, 1114, 3334, 4444\rangle$	3_3	
		$\langle 5_{74}, 2, 1134, 1334, 4444\rangle$	3_3	
		$\langle 5_{75}, 3, 0011, 0022, 2244\rangle$	4_{12}	
		$\langle 5_{76}, 3, 0011, 1222, 1244\rangle$	3_3	
		$\langle 5_{77}, 3, 0012, 0022, 1244\rangle$	4_{12}	
		$\langle 5_{78}, 3, 0111, 0222, 1244\rangle$	3_3	
		$\langle 5_{79}, 3, 0111, 1222, 1444\rangle$		
		$\langle 5_{80}, 3, 0111, 1224, 1244\rangle$		
		$\langle 5_{81}, 3, 1111, 1222, 1444\rangle$	4_{14}	2_1
		$\langle 5_{82}, 3, 1111, 1224, 1244\rangle$	4_{15}	2_1
		$\langle 5_{83}, 3, 1111, 1224, 4444\rangle$	2_1 4_{14}	4_{14}
		$\langle 5_{84}, 3, 1114, 1224, 1444\rangle$	2_1 4_{15}	4_{15}

Table 5.

Wasjberg hoops, hence they are the **0**-free reducts of the MV-chains that are usually denoted $\mathbf{MV}_1, \mathbf{MV}_2, \mathbf{MV}_3, \mathbf{MV}_4$. Likewise the algebras $2_1, 3_2, 4_4, 5_8$ are linearly ordered Brouwerian algebras, so they are reducts of the Gödel algebras $\mathbf{GA}_2, \mathbf{GA}_3, \mathbf{GA}_4, \mathbf{GA}_5$. The algebras $D_{3,2}$ and $D_{4,2} = D_4$ are two 4-element

RL var	FL var	Name, id, transformations	Sub	Hom
CDRL	DFL_e	$\langle E_3, 1, 2244, 3434, 4444\rangle$	4_{11}	
		$\langle E_4, 1, 2244, 3444, 4444\rangle$	4_9 4_{11}	
		$\langle E_5, 1, 2444, 3444, 4444\rangle$	4_9	
		$\langle E_6, 2, 0101, 0303, 1434\rangle$		
		$\langle E_7, 2, 0101, 0313, 1434\rangle$		
		$\langle E_8, 2, 0101, 0333, 1434\rangle$		
		$\langle E_9, 2, 0111, 1324, 1444\rangle$	4_{12}	
		$\langle E_{10}, 2, 0111, 1333, 1434\rangle$	4_{12}	
		$\langle E_{11}, 2, 0111, 1344, 1444\rangle$	4_{12}	
		$\langle E_{12}, 2, 1111, 1313, 1434\rangle$	4_{13} D_2	2_1
		$\langle E_{13}, 2, 1111, 1324, 1444\rangle$	4_{13} D_3	2_1
		$\langle E_{14}, 2, 1111, 1333, 1434\rangle$	4_{13} D_5	2_1
		$\langle E_{15}, 2, 1111, 1344, 1444\rangle$	4_{13} D_4	2_1
		$\langle E_{16}, 2, 1133, 3333, 3434\rangle$	2_1 3_3	3_3
		$\langle F_4, 1, 2022, 3234, 4244\rangle$	3_3	
		$\langle F_5, 1, 2134, 3334, 4444\rangle$	4_{11}	
		$\langle F_6, 1, 2222, 3234, 4244\rangle$	3_3	
		$\langle F_7, 1, 2224, 3234, 4444\rangle$	4_{11}	
		$\langle F_8, 1, 2334, 3334, 4444\rangle$	4_{11}	
		$\langle F_9, 1, 2444, 3444, 4444\rangle$	4_9	
DRL	DFL	$\langle E_{17}, 2, 0101, 1333, 1434\rangle$		
		$\langle E_{18}, 2, 0111, 0333, 1434\rangle$		
		$\langle E_{19}, 2, 1111, 3333, 3434\rangle$		
		$\langle E_{20}, 2, 1133, 1333, 1434\rangle$		
		$\langle F_{10}, 1, 2222, 3234, 4444\rangle$	3_3	
		$\langle F_{11}, 1, 2224, 3234, 4244\rangle$	3_3	

Table 6.

symmetric relation algebras that generate varieties covering the trivial variety (as seen in Figure 4).

The software to calculate the tables and diagrams was written in Python using the open-source computer algebra package Sage [8]. The subalgebras and homomorphic images were calculated with the help of the model finder package Mace4 [6], and the layout of the diagrams was improved by using the poset applet at `www.chapman.edu/~jipsen/posets.html`.

From the data in this note one can make a few observations. Even for finite simple residuated lattices it is unlikely that a good structure theory can be found. There do not seem to be many restrictions on the monoids that can be the reducts of finite residuated lattices. On the other hand for finite MV-algebras, BL-algebras, Heyting algebras and GBL-algebras, there does exist a good structure theory for the finite algebras: finite MV-algebras are direct products of finite MV-chains, finite Heyting algebras are finite distributive lattices expanded with a Heyting arrow, finite GBL-algebras are poset products of MV-chains [4], whence finite BL-algebras are tree products of MV-chains [2]. Perhaps a careful

RL var	FL var	Name, id, transformations	Sub	Hom
CRL	InFL_e	$\langle M_{1,1}, 1, 2022, 3214, 4244\rangle$	D_2 D_3	
		$\langle M_{2,3}, 1, 2022, 3244, 4244\rangle$	D_2 D_4	
		$\langle M_{3,1}, 1, 2202, 3033, 4234\rangle$	3_3	
		$\langle M_{4,1}, 1, 2314, 3124, 4444\rangle$	3_3	
		$\langle M_{5,3}, 1, 2344, 3444, 4444\rangle$	D_4	
		$\langle N_1, 1, 2002, 3023, 4234\rangle$	3_3	
		$\langle N_2, 1, 2002, 3033, 4234\rangle$	3_3	
		$\langle N_3, 2, 0111, 1344, 1444\rangle$	4_9 D_2	
		$\langle N_4, 2, 3144, 4344, 4444\rangle$	4_9	
		$\langle N_5, 3, 0111, 1224, 1444\rangle$	4_{10} D_2	D_2
		$\langle N_6, 3, 2114, 1224, 4444\rangle$	4_{10}	D_3
CRL	FL_e	$\langle M_6, 1, 2134, 3333, 4434\rangle$	D_3 D_5	
		$\langle M_7, 1, 2222, 3244, 4244\rangle$	D_4 D_5	
		$\langle M_8, 1, 2444, 3444, 4444\rangle$	D_4	
		$\langle N_7, 1, 2002, 3003, 4234\rangle$	D_2	
		$\langle N_8, 1, 2022, 3233, 4234\rangle$	D_2 D_5	
		$\langle N_9, 1, 2222, 3223, 4234\rangle$	D_5	
		$\langle N_{10}, 1, 2222, 3233, 4234\rangle$	D_5	
		$\langle N_{11}, 1, 2333, 3333, 4334\rangle$	D_5	
		$\langle N_{12}, 1, 2444, 3444, 4444\rangle$	D_4	
		$\langle N_{13}, 2, 0111, 1334, 1444\rangle$	4_{11} D_2	
		$\langle N_{14}, 2, 1111, 1334, 1444\rangle$	4_{11} D_5	
		$\langle N_{15}, 2, 1111, 1344, 1444\rangle$	4_9 D_5	
		$\langle N_{16}, 2, 3114, 1334, 4444\rangle$	4_{11}	
		$\langle N_{17}, 2, 4114, 1334, 4444\rangle$	4_{11}	
		$\langle N_{18}, 2, 4144, 4334, 4444\rangle$	4_{11} D_4	
		$\langle N_{19}, 2, 4144, 4344, 4444\rangle$	4_9 D_4	
		$\langle N_{20}, 3, 1011, 0222, 1244\rangle$	3_3	2_1
		$\langle N_{21}, 3, 1111, 1224, 1444\rangle$	4_{10} D_5	D_5
		$\langle N_{22}, 3, 4114, 1224, 4444\rangle$	4_{10} D_4	D_4
RL	FL	$\langle N_{23}, 1, 2002, 3233, 4234\rangle$	3_3	
		$\langle N_{24}, 1, 2022, 3033, 4234\rangle$	3_3	
		$\langle N_{25}, 2, 4114, 4334, 4444\rangle$	4_{11}	
		$\langle N_{26}, 2, 4144, 1334, 4444\rangle$	4_{11}	

Table 7.

examination of other finite residuated lattices will lead to further subvarieties where all finite members can be characterised.

Since the bottom element of a residuated lattice (if it exists) must always be a multiplicative zero, the identity element of a nontrivial residuated lattice cannot take the place of the bottom element. However there are other restrictions. The elements that are covered by the identity element need to generate a Boolean algebra under join and meet [9]. Hence there are no integral residuated lattices with the diamond M or pentagon N as lattice reduct. The data in these tables (and longer versions of them) can be used to discover other results of this form,

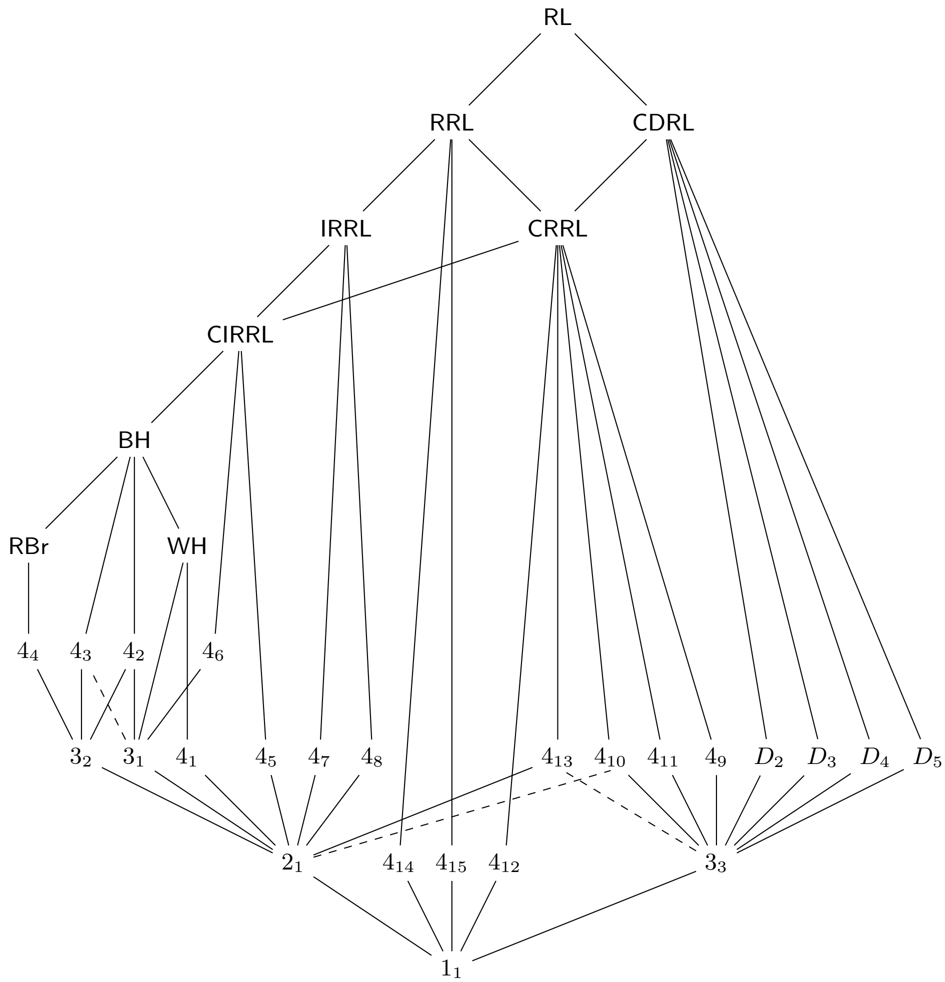

Fig. 2. HS-poset of residuated lattices with ≤ 4 elements; remove dotted lines to get the HS-poset of FL_o-algebras

where the noticeable absence of certain configurations in all finite algebras up to a certain size leads to the discovery of results that prove these configurations can never occur in a (finite) residuated lattice.

Linearly ordered residuated lattices have been studied in much more detail than subdirectly irreducible residuated lattices with a nonlinear or even nondistributive lattice reduct. This is largely because linear orders fits well with the algebraic theory of fuzzy logics. However the study of FL-algebras with weak forms of classical negation is also of interest in substructural logic, and such varieties are generally not determined by their linear members. Also varieties of

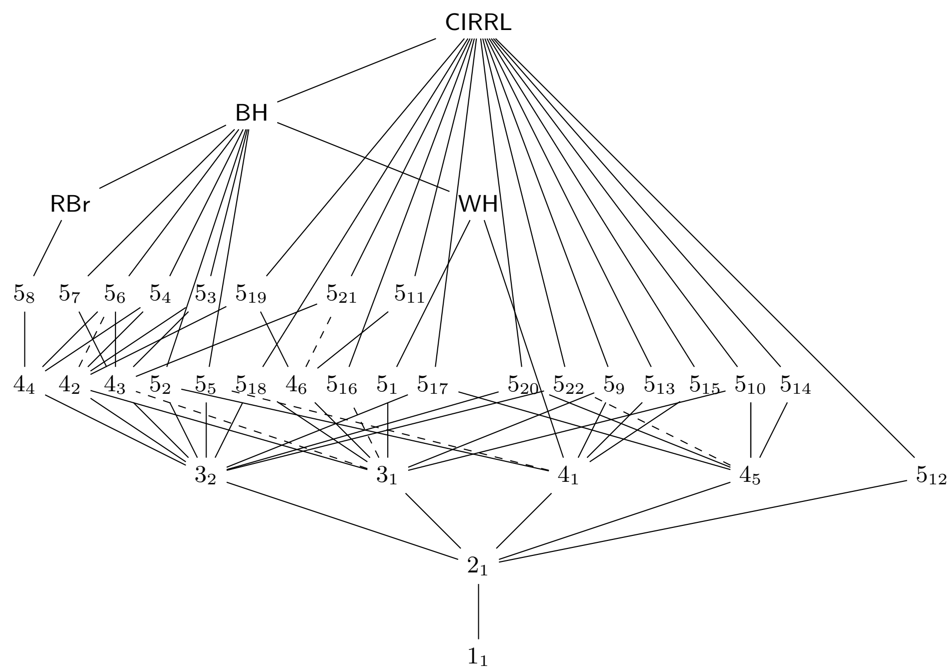

Fig. 3. HS-poset of CIRRL with ≤ 5 elements; remove dotted lines to get the HS-poset of RFL_{ew}-algebras

modular residuated lattices and residuated ortholattices are not representable, and studying their finite members may provide some insight or lead to conjectures and results that may clarify aspects of these varieties.

Lastly, it is currently not known whether the variety of residuated lattices has the amalgamation property. The finite algebras in this note could be a starting point for setting up specific V-formations and testing if an amalgam can be found in each case. If a general method can be developed from several such examples, this could lead to a proof of the amalgamation property, and if a counter example is found, then since the algebras involved are quite small, it is likely to be a minimal failure of the amalgamation property.

3 Conclusion

While it is relatively easy to enumerate residuated lattices and FL-algebras with more than 5 elements (up to about 10-12 elements depending on what subvariety one restricts to), it is not so simple to get good diagrams of the HS-posets for more than about 50 subdirectly irreducible algebras. The lattice of all subvarieties that are determined by FL-algebras of size ≤ 5 is isomorphic to the lattice of downsets of a poset that has 820 elements. The poset can be calculated, but

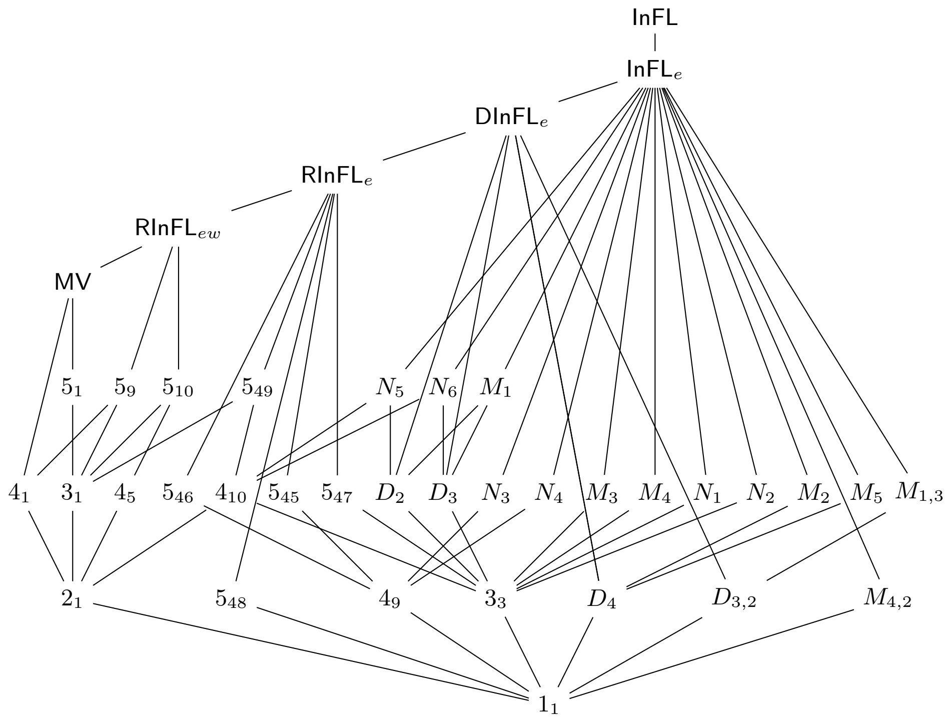

Fig. 4. HS-poset of InFL-algebras with ≤ 5 elements

it is unlikely that the number of downsets can be effectively determined since the poset is very wide and thus would have on the order of 2^{300} downsets.

To get manageable diagrams, we restricted to join irreducible subvarieties given by residuated lattices with ≤ 4 elements, subvarieties of commutative integral linear residuated lattices and subvarieties of involutive FL-algebras. The algebras and diagrams in this note can be used to investigate several interesting questions about residuated lattices and FL-algebras.

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